Sets, Relations and Functions

- Number of subsets of a set of n elements is 2^n.
- Number of proper subsets of a set of n elements is $2^n - 1$.
- $\{a, b\}$ and $\{b, a\}$ are two equal sets, but (a, b) and (b, a) are not equal ordered pairs.
- Universal relation and identity relation on non-empty set are always reflexive, symmetric and transitive.
- Identity relation on non-empty set is anti-symmetric.
- If $R : A \to B$ and $S : B \to C$, then $(RoS)^{-1} = S^{-1} o R^{-1}$.
- For two relations R and S, relations RoS and SoR may be empty relation.
- Let A and B are two non-empty finite sets and $f : A \to B$ is a function. This function will
  1. one-one, if $n(A) \le n(B)$
  2. onto, if $n(A) \ge n(B)$
  3. one-one onto, if $n(A) = n(B)$
- If A and B are two non-empty set, in which number of elements are m and n respectively, then
  1. number of elements in $A \times B = mn$
  2. number of relations from A to $B = 2^{mn}$
  3. number of functions from A to $B = n^m$
  4. number of one-one function $= \begin{cases} {}^nP_m, & \text{if } m \le n \\ 0, & \text{if } m > n \end{cases}$
  5. number of onto function
  $$= \begin{cases} 0, \text{ if } m < n \\ \sum_{r=1}^{m} (-1)^{n-r} \, {}^nC_r \, r^m, \text{ if } m \ge n \end{cases}$$
  6. number of one-one onto function from A to B
  $$= \begin{cases} n!, \text{ if } m = n \\ 0, \text{ if } m \ne n \end{cases}$$
- If A is a non-empty set in which number of elements is n, then
  1. number of elements in $A \times A = n^2$
  2. number of relations from A to $A = 2^{n^2}$
  3. number of functions from A to $A = n^n$
  4. number of one-one onto function from A to $A = n!$
  5. number of onto function from A to A
  $$= \sum_{r=1}^{n} (-1)^{n-r} \, {}^nC_r \, r^n$$
- If f is a function, then $\text{Dom}(f) = \text{Range}(f^{-1})$ and $\text{Range}(f) = \text{Dom}(f^{-1})$
- Number of onto functions of 2 elements from a finite set of n elements $(n \ge 2) = 2^n - 2$
- Product of two even functions is an even function and product of an even and an odd function is an odd function.
- Equal sets are always equivalent but equivalent sets may need not be equal set.
- If A has n elements, then power set $P(A)$ has 2^n elements.
- If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup A_3 \cup \ldots\ldots \cup A_n$.
- If $A_1, A_2, A_3, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$.
- $R - Q$ is the set of all irrational numbers.
- Let A and B be two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.
- The universal relation on a set A containing atleast two elements is not anti-symmetric, because if $a \ne b$ are in A, then a is related to b and b is related to a under the universal relation will imply that $a = b$ but $a \ne b$.
- The set $\{(a, a) : a \in A\} = D$ is called the diagonal line of $A \times A$. Then "the relation R in A is anti-symmetric, iff $R \cap R^{-1} \subseteq D$".
- The relation 'is congruent to' on the set T of all triangles in a plane is a transitive relation.
- If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- The inverse of an equivalence relation is an equivalence relation.

Complex Numbers

- If $\text{amp}(z) - \text{amp}(-z) = \begin{cases} \pi, & \text{if } \text{amp}(z) > 0 \\ -\pi, & \text{if } \text{amp}(z) < 0 \end{cases}$
- $\log_e(z) = \log_e|z| + i\ \text{amp}(z)$
- Triangle of vertices $P(z_1)$, $Q(z_2)$ and $R(z_3)$ will be equilateral, if

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

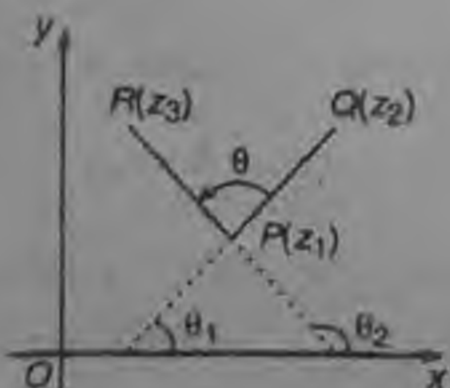
or $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

- If $\left|z + \frac{1}{z}\right| = a$, then

$$\text{maximum value of } z = \frac{a + \sqrt{a^2 + 4}}{2}$$

and $$\text{minimum value of } z = \frac{-a + \sqrt{a^2 + 4}}{2}$$

- The equation $|z - z_1|^2 + |z - z_2|^2 = k$ (where k is a real number) will represent a circle with centre at $\frac{1}{2}(z_1 + z_2)$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{1}{2}|z_1 - z_2|^2$.
- $(iz) = -i\bar{z}$, $\text{Re}(iz) = -\text{Im}(z)$, $\text{Im}(iz) = \text{Re}(z)$.
- If the complex numbers z_1 and z_2 are such that the sum $z_1 + z_2$ is a real number, then they are not necessarily conjugate complex.
- If z_1 and z_2 are two complex numbers such that the product z_1z_2 is a real number, then they are not necessarily conjugate complex.
- If z_1, z_2, z_3 are the affixes of the points P, Q, R respectively on Argand plane. Then angle between PQ and PR i.e., $\angle QPR$ is given by $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$.



$$\therefore \quad \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|(\cos\theta - i\sin\theta)$$

where $\theta = \angle QPR = \theta_2 - \theta_1$

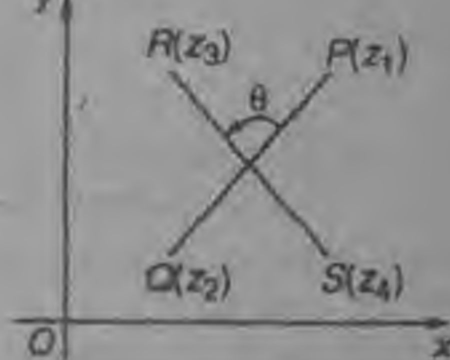
1. If z_1, z_2, z_3 are collinear, then angle $\theta = 0$, therefore $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

   Also z_1, z_2, z_3 are collinear, if $\arg(z_3 - z_1) = \arg(z_2 - z_1)$.
2. If z_1, z_2, z_3 are such that $PR \perp PQ$, then angle, $\theta = \frac{\pi}{2}$ and so $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary.

- If z_1, z_2, z_3, z_4 are affixes of four points P, Q, R, S respectively in the Argand plane, then SR is inclined to QP at an angle given by $\arg\left(\frac{z_3 - z_4}{z_1 - z_2}\right)$.

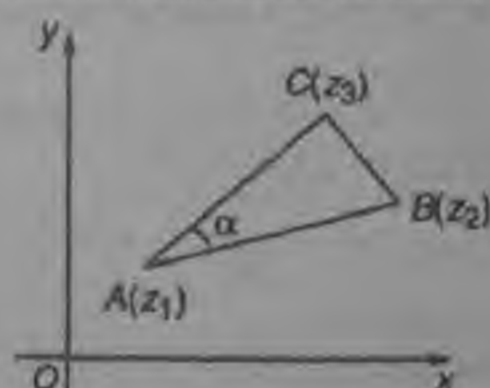
Therefore SR and QP are at right angles, provided

$$\arg\left(\frac{z_3 - z_4}{z_1 - z_2}\right) = \pm\frac{\pi}{2}$$



i.e., Provided $z_3 - z_4 = \pm ik(z_1 - z_2)$, where k is a non-zero number.

- If z_1, z_2, z_3 are affixes of the vertices of a triangle ABC described in anticlockwise sense, then

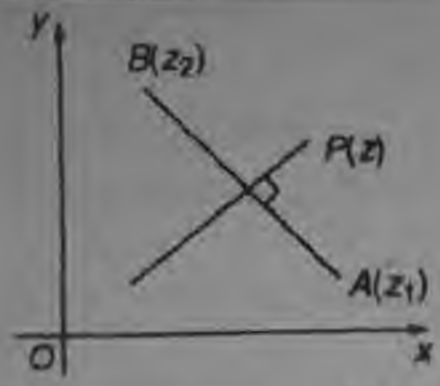


▲ Fig. 2.17

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{CA}{BA}(\cos\alpha + i\sin\alpha)$$

or $$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|e^{i\alpha} \quad (\text{where } \angle CAB = \alpha)$$

- Equation of perpendicular bisector of the join of the points A and B having z_1 and z_2 respectively as affixes, is given by



$$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$$

- The order relation is not defined on the set C of all complex numbers as it is not a complete ordered field. Thus the statements $z_1 > z_2$ and $z_1 < z_2$ have no meaning unless z_1 and z_2 both are purely real.
- For any $a, b \in R$
  1. $\sqrt{a+ib} + \sqrt{a-ib} = \sqrt{2(\sqrt{a^2+b^2}+a)}$
  2. $\sqrt{a+ib} - \sqrt{a-ib} = i\sqrt{2(\sqrt{a^2+b^2}-a)}$
- The one and only one case in which

$$|z_1| + |z_2| + \ldots + |z_n| = |z_1 + z_2 + \ldots + z_n|$$

is that the numbers $z_1, z_2, \ldots, z_n$ have the same amplitude.

The sum and product of two complex numbers are real simultaneously if and only if they are conjugate to each other.

- If three points z_1, z_2, z_3 connected by relation $az_1 + bz_2 + cz_3 = 0$ where $a + b + c = 0$, then the three points are collinear.
- Complex numbers z_1, z_2, z_3 are the vertices of A, B, C respectively of an isosceles right angled with right angled at C, then
$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$$
- The complex numbers z_1, z_2, z_3 be three vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.
- If z is a complex number, then e^z is periodic.
- If three complex numbers are in AP, then they lie on a straight line in the complex plane.

Quadratic Equations

- If b is of opposite sign as compared to a and c, then both roots of $ax^2 + bx + c = 0$ are positive.
- If a, b, c are all of same sign, then both roots of $ax^2 + bx + c = 0$ are negative.
- If there is no term containing coefficient of x, then both the roots of the equation $ax^2 + bx + c = 0$ are equal in magnitude but opposite in sign.
- If a and c are of opposite signs, then both the roots of the equation are of opposite sign.
- If the roots of $ax^2 + bx + c$ are reciprocal to each other, then $c = a$.
- If $f(\alpha) = 0$ and $f'(\alpha) = 0$, then α is repeated roots of $f(x) = 0$ and $f(x) = a(x - \alpha)^2$
$$\Rightarrow \qquad \alpha = -\frac{b}{2a}$$
- For quadratic equation
  (a) one root is zero, if $c = 0$
  (b) both roots are zero, if $b = c = 0$
- Ratio of roots of quadratic equation $ax^2 + bx + c = 0$ is $m : n$, then
$$nmb^2 = (m+n)^2 c.$$
- If one root of $ax^2 + bx + c = 0$ is nth power of another root, then
$$(ac^n)^{1/n+1} + (a^n c)^{1/n+1} + b = 0$$
- If one root of $ax^2 + bx + c = 0$ is n times of another root, then
$$nb^2 = ac(n+1)^2$$
- If α and β are the roots of $ax^2 + bx + c = 0$, then $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ will be the roots of $cx^2 + bx + a = 0$.
- If roots of $ax^2 + bx + c = 0$ are reciprocal of roots of $a'x^2 + b'x + c' = 0$, then
$$(cc' - aa')^2 = (ba' - cb')(ab' - bc')$$
- If α be the common roots to $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$, then
$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \text{ or } \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$
- If both roots are common of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
- If α and β are roots of equation $ax^2 + bx + c = 0$, then equation
  (a) whose roots are $\alpha \pm A, \beta \pm A$ is
  $$a(x \mp A)^2 + b(x \mp A) + c = 0$$
  (b) whose roots are $A\alpha, A\beta$ is $ax^2 + Abx + A^2c = 0$
  (c) whose roots are $\frac{\alpha}{A}, \frac{\beta}{A}$ is $aA^2x^2 + bAx + c = 0$
  (d) whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $cx^2 + bx + a = 0$
- If in $ax^2 + bx + c = 0$, $a > 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c$ is always positive.
- If in $ax^2 + bx + c$, $a < 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c$ is always negative.

- If the sum of coefficients of polynomial equation $a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = 0$, then $x = 1$ is always a root of equation.
- In quadratic equation $ax^2 + bx + c = 0$, if $a + b + c = 0$, then roots of equation will be 1 and $\frac{c}{a}$. And if $a - b + c = 0$, then roots of equation will be -1 and $-\frac{c}{a}$.
- If $a_1, a_2, a_3, \ldots, a_n$ are positive, then the minimum value of $(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$ will be n^2.
- If the sum of roots of $ax^2 + bx + c = 0$ is equal to the sum of their reciprocals, then ab^2, bc^2, ca^2 are in AP or $\frac{b}{c}, \frac{a}{b}, \frac{c}{a}$ will be in AP.
- $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right], \forall n \in N$
- $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = nx$
- $x = [x] + (x)$
- (a) $[-x] = -[x], x \in I$

  (b) $[-x] = -[x] - 1, x \notin I$

Inequalities

- If $x_1, x_2, \ldots, x_n$ are n positive variables such that $x_1 + x_2 + \ldots + x_n = c$ (constant), then the product $x_1 x_2 \ldots x_n$ is greatest when $x_1 = x_2 = \ldots = x_n = \frac{c}{n}$ and the greatest value is $\left(\frac{c}{n}\right)^n$.
- If $x_1, x_2, \ldots, x_n$ are positive variables such that $x_1 x_2 \ldots x_n = c$ (constant), then the sum $x_1 + x_2 + \ldots + x_n$ is least when $x_1 = x_2 = \ldots = x_n = c^{1/n}$ and the least value of the sum is $n(c^{1/n})$.
- If $x_1, x_2, \ldots, x_n$ are variables and $m_1, m_2, \ldots, m_n$ are positive real numbers such that $x_1 + x_2 + \ldots + x_n = c$ (constant), then $x_1^{m_1} \cdot x_2^{m_2} \ldots x_n^{m_n}$ is greatest when

$$\frac{x_1}{m_1} = \frac{x_2}{m_2} = \ldots = \frac{x_n}{m_n} = \frac{x_1 + x_2 + \ldots + x_n}{m_1 + m_2 + \ldots + m_n}$$

- If $x_1 + x_2 + \ldots + x_m = c$, then
  1. $x_1^m + x_2^m + \ldots + x_n^m$ will be minimum, if $m \le 0$ or $m \ge 1$.
  2. $x_1^m + x_2^m + \ldots + x_n^m$ will be maximum, if $0 < m < 1$ and value will be $n^{1-m} c^m$ when $x_1 = x_2 = \ldots = x_n$.
- If $x_1^m + x_2^m + \ldots + x_n^m = c$, then
  1. $x_1 + x_2 + \ldots + x_m$ will be maximum, if $m \le 0$ or $m \ge 1$.
  2. $x_1 + x_2 + \ldots + x_n$ will be minimum if $0 < m < 1$ and value will be $n^{1-\frac{1}{m}} - k^{\frac{1}{m}}$ when $x_1 = x_2 = \ldots = x_n$.

Sequence and Series

- If a, b, c are in AP, GP or HP according as $\frac{a-b}{b-c} = \frac{a}{a}, \frac{a}{b}$ or $\frac{a}{c}$.
- If $a, b, c, d, \ldots$ are in GP, they are also in continued proportion i.e.,

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots = \frac{1}{r} \text{ (say)}.$$

- If a, b, c are in AP, then x^a, x^b, x^c will be in GP $(x \ne \pm 1)$.
- If an AP consists of n (odd terms) and its middle term is m, then the sum of the AP is mn.
- If pth, qth and rth terms of a GP are in GP, then p, q, r are in AP.
- If first term of a GP of n terms is a and last term is L Then the product of all the terms of the GP is $(aL)^{n/2}$.
- If a, b, c are in AP, as well as in GP, then $a = b = c$.
- T_n and S_n of any AP are of the form $an + b$ and $an^2 + bn$ respectively where a and b are constants.
- T_n and S_n of the series whose difference of terms i.e., $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$ form an AP are of the type $an^2 + bn + c$ and $an^3 + bn^2 + cn$ respectively, where a, b, c are constants.
- If there be n quantities in GP whose common ratio is r and S_m denotes the sum of the first m terms, then the sum of their products taken by two is $\frac{r}{r+1} S_m S_{m-1}$.
- If nth term of a series is $T_n = an^2 + bn + c$, then sum of its n terms is given by $S_n = a\Sigma n^2 + b\Sigma n + cn$. In general $S_n = \Sigma T_n$.

- If pth term of an AP be q and qth term be p, then $(p+q)$th term will be zero and its nth term will be $(p+q-n)$.
- If pth term of AP is $\frac{1}{q}$ and qth term is $\frac{1}{p}$, then its (pq)th term will be 1 and $S_{pq}=\frac{1}{2}(pq+1)$.
- If in AP, $S_p=q$ and $S_q=p$, then
$$S_{(p+q)}=-(p+q)$$
- The equation having a and b as its roots is
$$x^2-2Ax+G^2=0$$
- If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is
$$x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$$
- Recognising the type of the infinite series :
  (a) Each of the binomial, exponential and logarithmic series has infinite terms.
  (b) Series involving n ! s in the denominators of terms are generally binomial or exponential series. But in a binomial series the number of factors in the numerators (other than the power of a fixed number) of terms goes on increasing.
  (c) Logarithmic series do not contain n ! s in the denominators of terms. The term contain $\frac{1}{n}$ s.

Binomial Theorem

- The number of terms in $(x+y+z)^n$ is ${}^{n+2}C_2$.
- Sum of coefficients in the expansion of $(a+bx+cx^2)^n$ is $(a+b+c)^n$.
- Coefficient of $x^{n_1}\cdot y^{n_2}\cdot z^{n_3}$ in the expansion of $(x+y+z)^n$ is $\frac{n!}{n_1!\,n_2!\,n_3!}$, where $n=n_1+n_2+n_3$.
- General term in the expansion of $(a_1+a_2+\ldots+a_k)^n$ is $\frac{n!}{r_1!\,r_2!\ldots r_k!}x_1^{r_1}\cdot x_2^{r_2}\ldots x_k^{r_k}$ and total number of terms in expansion is ${}^{n+k-1}C_{k-1}$.
- Coefficient of x^m in $(1+x^r)^n$ is 0, if m is not a multiple of r.
  e.g., Coefficient of x^{1000} in the expansion of $(1+x^3)^{2000}$ is zero, since 1000 is not a multiple of 3.
- If value of x is so small that on leaving square and higher powers of x
$$(1+x)^n=1+nx.$$
- Coefficient of x^{n-1} in the expansion of $(x-1)(x-2)\ldots(x-n)$ is $-\frac{n(n+1)}{2}$.
- Coefficient of x^{n-1} in the expansion of $(x+1)(x+2)\ldots(x+n)$ is $\frac{n(n+1)}{2}$.
- The number of terms in $(x+a)^n+(x-a)^n$ will be $\frac{n+2}{2}$, if n is an even number and will be $\frac{n+1}{2}$, if n is an odd number.

Determinant

- If $A=B+C$, then it is not necessary that $\det(A)=\det(B)+\det(C)$.
- If A is a square matrix of order $n\times n$, then
$$\det(kA)=k^n(\det A)$$
- If A, B, C are three square matrix such that ith row of A is equal to the sum of ith row of B and C and remaining rows of A, B and C are same, then
$$\det(A)=\det(B)+\det(C)$$
- **Symmetric determinant :** A determinant is called symmetric determinant, if for its every element $a_{ij}=a_{ji}\ \forall\ i, j$.

e.g., $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2$

- **Skew-symmetric determinant :** A determinant is called skew symmetric determinant, if for its every element $a_{ij}=-a_{ji}\ \forall\ i, j$

e.g., $\begin{vmatrix} 0 & b & c \\ -b & 0 & a \\ -c & a & 0 \end{vmatrix}=0$

- 1. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=-(a^3+b^3+c^3-3abc)$

2. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

4. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)$

$(ab+bc+ca)$

- If $\Delta_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a, b, c, a_1, b_1, c_1 are constant, then

1. $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

2. $\prod_{r=1}^{n} \Delta_r = \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

- $\det(A^n) = (\det A)^n$, where n is a positive integer.
- Determinant of a diagonal matrix is product of all diagonal elements.
- Determinant of a triangular matrix is also product of diagonals.
- Determinant of an identity matrix is 1.

Matrices

- $A + A'$ is always a symmetric matrix.
- $A - A'$ is always a skew-symmetric matrix.
- AA' is always a symmetric matrix.
- The matrix $B'AB$ is symmetric or skew-symmetric matrix according as A is symmetric or skew-symmetric.
- All positive integral powers of a symmetric matrix are symmetric.
- Positive odd integral powers of skew-symmetric matrix are skew-symmetric and positive even integral powers of skew-symmetric matrix are symmetric.
- The inverse of a symmetric matrix is symmetric.
- The inverse of a diagonal matrix is diagonal.
- A diagonal matrix is both an upper triangular and a lower triangular.
- If A and B are symmetric matrices of order n, then ABA is a symmetric matrix.
- If A is a non-singular matrices, then $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Every orthogonal matrix is invertible.
- Every invertible matrix is not necessary orthogonal.
- If A is an orthogonal matrix, then A^{-1} is also orthogonal.
- Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.
- If A and B are two symmetric (or skew-symmetric) matrices of same order, then $A + B$ is also symmetric (or skew-symmetric).
- Every square matrix can be uniquely expressed as the sum of symmetric matrix and a skew-symmetric matrix.
- The number of non-zero rows of a matrix given in the Echelon form is its rank.
- If A is a skew-hermitian matrix, then kA is also skew- hermitian for any real number k.
- If A and B are skew-hermitian matrices of same orders, then $\lambda_1 A + \lambda_2 B$ is also skew-hermitian for any real number as λ_1, λ_2 etc.
- If A and B are hermitian matrices of same order, then $AB - BA$ is skew-hermitian.
- If A is any square matrix, then $A - A^*$ is a skew-hermitian matrix.
- Every square matrix can be uniquely represented as the sum of a hermitian and a skew-hermitian matrices.
- If A is a skew-hermitian matrix, then iA is a hermitian matrix.
- If A is a skew-hermitian matrix, then $\bar{A}$ is also skew-hermitian matrix.
- If A is hermitian matrix, then kA is also hermitian for any real number k.
- If A and B are hermitian matrices of same order, then $\lambda_1 A + \lambda_2 B$, also hermitian for any real number as λ_1, λ_2.
- If A be any square matrix, then AA^* and A^*A are also hermitian.
- If A and B are hermitian, then AB is also hermitian, iff $AB = BA$.
- If A is a hermitian matrix, then $\bar{A}$ is also hermitian.

- If A and B are hermitian, matrices of same order, then $AB + BA$ is also hermitian.
- If A is a square matrix, then $A + A^*$ is a hermitian matrix.
- Any square matrix can be uniquely expressed as $A + iB$, where A and B are hermitian matrices.
- If A and B are idempotent matrices, then AB is an idempotent, iff $AB = BA$.
- If A and B are idempotent matrices, then $A + B$ is an idempotent, iff $AB = BA = O$.
- If A is an idempotent and $A + B = I$, then B is an idempotent and $AB = BA = O$.
- Diagonal $(1, 1, 1, \ldots, 1)$ is an idempotent matrix.
- If $AB = A$ and $BA = B$, then $A^2 = A$, $B^2 = B$.
- If l_1, l_2, l_3 ar direction cosines, then $\begin{bmatrix} l_1^2 & l_1l_2 & l_1l_3 \\ l_1l_2 & l_2^2 & l_2l_3 \\ l_3l_1 & l_3l_2 & l_3^2 \end{bmatrix}$ is an idempotent as $|\Delta|^2 = 1$.
- If A, B, C are square matrices of the same order such that ith (or row) of A is the sum of ith columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $\det A = \det B + \det C$.

Permutations and Combinations

- Out of n non-concurrent and non-parallel straight lines the number of point of intersections are ${}^nC_2$.
- The number of straight lines passing through n points $= {}^nC_2$.
- The number of straight lines passing through n points out of which m are collinear $= {}^nC_2 - {}^mC_2 + 1$.
- In a polygon, the total number of diagonals out of n points (no three points are collinear) $= \dfrac{n(n-3)}{2}$.
- Number of triangles formed by joining n points is ${}^nC_3$.
- Number of triangles formed by joining n points out of which m are collinear, are ${}^nC_3 - {}^mC_3$.
- The number of parallelogram in two system of parallel lines (when Ist set contains m parallel lines and 2nd set contains n parallel lines) $= {}^nC_2 \times {}^mC_2$ and number of squares
$$= \sum_{r=1}^{m-1} (m-r)(n-r); (m < n).$$
- Number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.
- In a rectangle of $n \times p$ $(n < p)$ number of rectangles of any size is $\dfrac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n} (n+1-r)(p+1-r)$.
- n straight lines are drawn in a plane such that no two lines are parallel and no three lines are concurrent. Then the number of parts into which these lines divide the plane, is equal to $1 + \Sigma n$.
- The sum of the digits in the unit place of all numbers formed with the help of $a_1, a_2, \ldots, a_n$ taken all at a time is $(n-1)!(a_1 + a_2 + \ldots + a_n)$ [repetition of digits is not allowed].
- The sum of all digits numbers that can be formed using the digits $a_1, a_2, \ldots, a_n$ is $(n-1)!$ $(a_1 + a_2 + \ldots + a_n)\dfrac{(10^n - 1)}{9}$ (repetition of digits is not allowed).
- $n! + 1$ is not divided by any number between 2 and n.
- Number of terms in $(a_1 + a_2 + \ldots + a_n)^m$
$$= {}^{n+k-1}C_{k-1}, \text{ where } k \text{ is number of terms.}$$
- If in any party n persons are present, then total number of hand shakes $= {}^nC_2$.

Probability

- 1. $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$
  2. $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$
  3. $P(A) = P(A \cap B) + P(A \cap \bar{B})$
  4. $P(B) = P(B \cap A) + P(B \cap \bar{A})$
- If A and B are two events such that $B \neq \phi$, then $P(A/B) + P(\bar{A}/B) = 1$

1. Probability of inserting all n letters in right addressed envelopes is $\dfrac{1}{n!}$.
2. Probability of keeping at least one letter in wrong envelope $= 1 - \dfrac{1}{n!}$

3. Probability of keeping all the n letters in wrong envelopes $= \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$
4. Probability of keeping at least one letter in right addressed envelope $= 1 - p$.

- **Multinomial theorem :** If a dice has m faces marked 1, 2, 3, ..., m and if such n dice are thrown, then the probability that the sum of the numbers on the upper face is equal to r is given by the coefficient of x^r in $\frac{(x + x^2 + \ldots + x^m)^n}{m^n}$.
- **Mathematical expectation :** If corresponding probabilities values of X of discrete random variable $x_1, x_2, x_3, \ldots, x_n$ are $p_1, p_2, \ldots, p_n$, then mathematical expectation of X will be

$$E(X) = \sum_{i=1}^{n} x_i p_i, \qquad \text{where } \sum_{i=1}^{n} p_i = 1$$

i.e., $\quad E(X) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n,$

where $\quad p_1 + p_2 + \ldots + p_n = 1.$

- When two dice are thrown, then the number of ways of r to be sum of upper faces, will be
  1. $(r-1)$, if $2 \le r \le 7$
  2. $(13-r)$, if $8 \le r \le 12$
- When three dice are thrown, then the number of ways of r to be sum of upper faces, will be
  1. $\frac{(r-1)(r-2)}{2}$, if $1 \le r \le 8$
  2. $\frac{(19-r)(20-r)}{2}$, if $13 \le r \le 18$
  3. 25, if $r = 9$ or 12
  4. 27, if $r = 10$ or 11.
- If A and B are two finite sets and, if one mapping is taken from into mappings, then
  1. Probability of mapping to be one-one $= \frac{{}^{n(B)}P_{n(A)}}{\{n(B)\}^{n(A)}}$
  2. Probability of mapping to be many one $= 1 - \frac{{}^{n(B)}P_{n(A)}}{\{n(B)\}^{n(A)}}$
  3. Probability of constant mapping $= \frac{n(B)}{\{n(B)\}^{n(A)}}$.
  4. Probability of mapping to be one-one onto $= \frac{n(A)!}{\{n(B)\}^{n(A)}}$, if $n(A) = n(B)$

Properties of Triangle

- $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{s-c}{s}$ and $\cot\frac{A}{2}\cot\frac{B}{2} = \frac{s}{s-c}$
- $\tan\frac{A}{2} + \tan\frac{B}{2} = \frac{c}{s}\cot\frac{C}{2} = \frac{c}{\Delta}(s-c)$
- $\tan\frac{A}{2} - \tan\frac{B}{2} = \frac{a-b}{\Delta}(s-c)$
- $\cot\frac{A}{2} + \cot\frac{B}{2} = \frac{c}{(s-c)}\cot\frac{C}{2}$
- In any quadrilateral $ABCD$,

$$\sin(A+B) + \sin(C+D) = 0$$

and $\quad \cos(A+B) = \cos(C+D)$

- $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$
- $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - (a^2 + b^2 + c^2)$
- $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$

and $\quad r_1 r_2 r_3 = r^2\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^2$

- $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$
- If p_1, p_2, p_3 are perpendicular's drawn from the vertices to opposite side of the triangle, then

(a) $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$

(b) $p_1 p_2 p_3 = \frac{(abc)^2}{8R^3}$

(c) $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$

- Orthocentre of the triangle is the incentre of the pedal triangle.
- If I_1, I_2 and I_3 be the centres of escribed circles which are opposite to A, B and C respectively and I is the centre of incircle, then triangle ABC is the pedal triangle of the triangle $I_1I_2I_3$ and I is the orthocentre of the triangle $I_1I_2I_3$.
- The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1 : 2.
- Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as nine point circle.

- Circumcentre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the orthocentre.
- Circumradius of pedal triangle is $\frac{R}{2}$.
- Area of pedal triangle is $2\Delta \cos A \cos B \cos C$.
- Sum of the opposite angles of a cyclic quadrilateral is $180°$.
- In a cyclic quadrilateral sum of the products of the opposite sides is equal to the product of the diagonals. This is known as Ptolemy's theorem. In any cyclic quadrilateral $ABCD$,

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

- If sum of the opposite sides of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.
- Circumradius of a cyclic quadrilateral,

$$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$$

- Sum of interior angles of polygon of n sides is $(n-2)\pi$ and each angle is $(n-2)\frac{\pi}{n}$.
- Sum of external angles of a polygon is 360.
- In a regular polygon, centroid, circumcentre and incentre are coincide.
- Area of cyclic quadrilateral

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

And $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab+cd)}$

Inverse Trigonometric Functions

- If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then $xy + yz + zx = 1$
- If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then $x + y + z = xyz$
- If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$, then

$$x^2 + y^2 + z^2 + 2xyz = 1$$

- If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, then

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

- If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$, then $xy + yz + zx = 3$
- If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then

$$x^2 + y^2 + z^2 + 2xyz = 1$$

- If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then

$$xy + yz + zx = 3$$

- If $\sin^{-1} x + \sin^{-1} y = \theta$, then $\cos^{-1} x + \cos^{-1} y = \pi - \theta$
- If $\cos^{-1} x + \cos^{-1} y = \theta$, then $\sin^{-1} x + \sin^{-1} y = \pi - \theta$
- If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{2}$, then $xy = 1$
- If $\cot^{-1} x + \cot^{-1} y = \frac{\pi}{2}$, then $xy = 1$
- If $\cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b} = \theta$, then

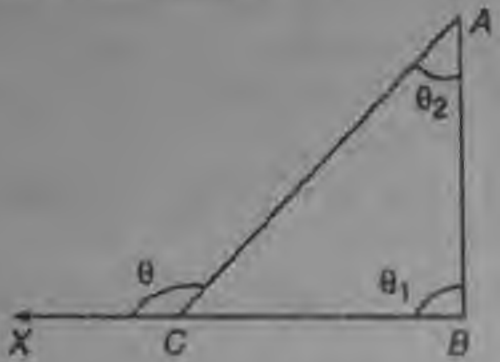
$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$$

Heights and Distances

- Angle of elevation and angle of depression are always acute angles.
- Any perpendicular line to a plane is perpendicular to all lines lying in the plane.
- In isosceles triangle, the median is perpendicular to the base.

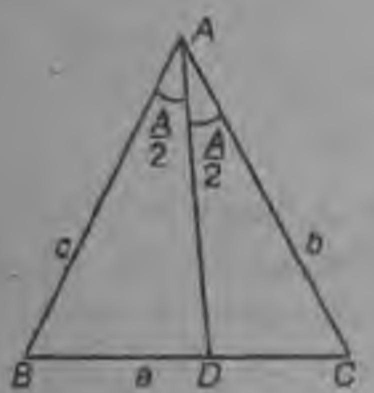
In Fig. 17.14 ΔABC is isosceles and $AD \perp BC$.

- In similar triangle, the corresponding sides are proportional.
- The exterior angle of a triangle is equal to the sum of interior opposite angles.



In ΔABC, $\theta = \theta_1 + \theta_2$

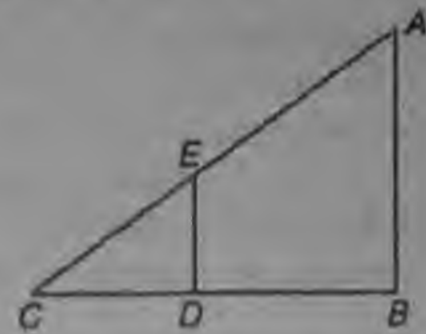
- In a triangle, the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.



In Fig. 17.16, AD is bisector of $\angle A$.

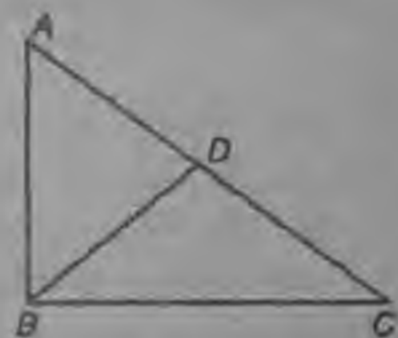
$$\frac{BD}{DC} = \frac{c}{b}$$

- In a triangle ABC, if $DE \parallel AB$, then



$$\frac{AB}{DE} = \frac{BC}{DC} = \frac{AC}{EC}$$

- In a right angled triangle ABC, the mid point D of hypotenuse AC is equidistant from its vertices A, B and C.



i.e., $AD = BD = CD$

- Angles in the same segment of a circle are equal.
- The angle subtended at the centre by an arc is double of angle subtended at any point of the remaining arc by the same arc.

Rectangular Cartesian Co-ordinates

- If (x_1, y_1), (x_2, y_2) are the ends of hypotenuse of a right angled isosceles triangle, then the third vertex is given by $\left(\frac{x_1 + x_2 \pm (y_1 - y_2)}{2}, \frac{y_1 + y_2 \pm (x_1 - x_2)}{2}\right)$.
- Given two vertices (x_1, y_1) and (x_2, y_2) of an equilateral triangle, then its third vertex is given by $\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp \sqrt{3}(x_1 - x_2)}{2}\right)$
- Circumcentre of the right angled triangle ABC right angled at A is $\frac{B + C}{2}$.
- Orthocentre of the right angled triangle ABC, right angled at A is A.
- x-axis divides the line segment joining (x_1, y_1), (x_2, y_2) in the ratio $-y_1 : y_2$ and y-axis divides the same line segment in the ratio $-x_1 : x_2$.
- If $ABCD$ is a parallelogram, then $D = A - B + C$.
- If D, E, F are the mid points of the sides BC, CA, AB of (ΔABC) then

respectively, $$A = E + F - D$$ $$B = F + D - E$$ $$C = D + E - F$$

- If D, E, F are the mid points of sides BC, CA, AB of a ΔABC, respectively, then the centroid of ΔABC = centroid of ΔDEF.
- Orthocentre, centroid, circumcentre of triangle are collinear. Centroid divides the line segment joining orthocentre and circumcentre in the ratio 2 : 1.
- The circumcentre of a right angled triangle is the mid point of hypotenuse.
- In an equilateral triangle orthocentre, centroid, circumcentre, incentre coincide.
- A triangle is isosceles, if any two of its medians are equal.

- Area of n-sided regular polygon is $\frac{1}{4} na^2 \cot\left(\frac{\pi}{n}\right)$.
- Area of cyclic quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.
- If the algebraic sum of the perpendicular distances from the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of the triangle ABC to a variable line is zero, then line passes through the centroid of the triangle ABC.
- If a triangle having integral co-ordinates, then it cannot be equilateral triangle.
- Centroid of a triangle is same as that of a triangle formed by joining mid points of the side of the triangle.
- If ΔABC is right angle at C, then radius of inscribed circle is $\frac{a+b-c}{2}$, where a, b, c are the lengths of respective sides.
- If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$, then two triangles with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are equal in area.
- If $p_1 \neq p_2 \neq p_3$, then three points $(p_r, 2ap_r + p_r^3)$ $(r = 1, 2, 3)$ are collinear, if $\sum_{r=1}^{r=3} p_r = 0$, i.e., $p_1 + p_2 + p_3 = 0$.
- If the vertices of a triangle are $(ap_1 p_2, a(p_1 + p_2))$, $(ap_2 p_3, a(p_2 + p_3))$ and $(ap_1 p_3, a(p_1 + p_3))$ then orthocentre of the triangle is the point $[-a, a(p_1 + p_2 + p_3 + p_1p_2p_3)]$ and orthocentre lies on x-axis, if $p_1 + p_2 + p_3 = -p_1p_2p_3$, then co-ordinate of orthocentre be $(-a, 0)$.
- Diagonals of a quadrilateral formed by the lines $ax + by + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular, if $a^2 + b^2 = a'^2 + b'^2$.
- If x_1, x_2, x_3 and y_1, y_2, y_3 are in GP with same ratio.
- Then set of points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear. As the points are collinear so area of triangle formed with vertices A, B, C is zero.
- If x_1, x_2, x_3 are in GP with common ratio R and y_1, y_2, y_3 are also in GP, with common ratio r, then area of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is given by $\frac{1}{2} x_1 y_1 (R-1)(r-1)(R-r)$.
- If sides of an equilateral triangle is a and vertices are given by $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{2} a^2.$$

Straight Lines and Pair of Straight Lines

- The foot of the perpendicular (h, k) from (x_1, y_1) to the line $ax + by + c = 0$ is given by

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}.$$

- The image or (reflection) of the point (x_1, y_1) in the line $ax + by + c = 0$ is (h, k) where $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$.
- Area of rhombus formed by $ax \pm by \pm c = 0$ is $\left|\frac{2c^2}{ab}\right|$.
- Area of parallelogram formed by the lines

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0,$$
$$a_1x + b_1y + d_1 = 0$$

and $a_2x + b_2y + d_2 = 0$ is $\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$.

- A straight line is such that the algebraic sum of the perpendicular drawn upon it from any number of fixed point is zero. Then the line always passes through a fixed point.
- If the vertices of a triangle have integral co-ordinates, then the triangle can't be equilateral.
- If the direction of the two sides of a triangle is fixed and length of the third side is constant and is sliding between these sides, then locus of the orthocentre of the triangle is circle.
- Orthocentre of a triangle will lie inside, if triangle is acute and in case of obtuse triangle orthocentre will lie outside and in case of right triangle orthocentre is the vertex at which it is right angled.
- In the normal form of a straight line, the sum of the squares of the coefficient of x and y is equal to one.
- Three or more straight lines are said to be concurrent lines, if they meet in a point.

- Lines represented by $ax^2 + 2hxy + by^2 = 0$ are mutually perpendicular, iff $a + b = 0$.
- Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident, iff $h^2 - ab = 0$, then $(ax^2 + 2hxy + by^2)$ is a perfect square.
- Lines perpendicular to lines represented by $ax^2 + 2hxy + by^2 = 0$ are $bx^2 - 2hxy + ay^2 = 0$. For perpendicular pairs interchange the coefficient of x^2 and y^2 and change the sign of xy.
- If $a = b$, then angle bisectors of $ax^2 + 2hxy + by^2 = 0$ are $x^2 - y^2 = 0$.
- If $h = 0$, then the angle bisectors of $ax^2 + 2hxy + by^2 = 0$ are $xy = 0$.
- If $a + b = 0$, then bisectors are always perpendicular to each other.
- Point of intersection of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$ or $\left(\sqrt{\frac{f^2 - bc}{h^2 - ab}}, \sqrt{\frac{g^2 - ca}{h^2 - ab}}\right)$.
- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the equation of lines through the origin and parallel to them is $ax^2 + 2hxy + by^2 = 0$.
- Pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle.
- If the two lines given by the pair of straight line $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on y-axis, then $2fgh - bg^2 - ch^2 = 0$.
- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then the square of the distance of their point of intersection from origin is $\frac{c(a + b) - f^2 - g^2}{ab - h^2}$
- If $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ form the sides of a triangle, then the area of this triangle is $\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$
- If $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ represent the sides of a triangle having the point (x_1, y_1) as its centroid, then $\frac{x_1}{bl - hm} = \frac{y_1}{am - bl} = \frac{2}{b(bl^2 - 2hlm + am^2)}$
- The condition for the lines $ax^2 + 2hxy + by^2 = 0$ and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ form a rhombus, is $(a - b)fg + h(f^2 - g^2) = 0$.
- If the lines joining the origin to the points of intersection of the curves $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x = 0$ are orthogonal, then
$$g_1(a_2 + b_2) = g_2(a_1 + b_1).$$

Circles and System of Circles

- $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, if $a = b$, i.e., coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.
- The centre of a circle is mid point of the diameter and radius is half of the length of diameter.
- Number of tangents: In general two real tangents can be drawn from a point outside a circle.
- The point 'θ' on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.
- A line touches a circle, if the length of the perpendicular from the centre is equal to the radius of the circle.
- A line intersects a given circle at two distinct real points, if the length of the perpendicular from the centre is less than the radius of the circle.
- A line does not intersect a circle, if the length of the perpendicular from the centre is greater than the radius of the circle.
- Length of the intercept cut off from the line $y = mx + c$ by circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2(1 + m^2) - c^2}{1 + m^2}}$.
- Condition of tangency of the line $y = mx + c$ to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$ i.e., $c = \pm a\sqrt{(1 + m^2)}$.
- Point of contact for tangent $y = mx + a\sqrt{1 + m^2}$ to the circle $x^2 + y^2 = a^2$ is $\left(\pm\frac{am}{\sqrt{1 + m^2}}, \mp\frac{a}{\sqrt{1 + m^2}}\right)$.
- Power of a point P w.r.t. to a circle is PT^2, if PT is tangent of the circle.
- If C_1, C_2, C_3 represents centre of circle $x^2 + y^2 + 2h_ix - c^2 = 0$ $(i = 1, 2, 3)$ and O is the origin, then OC_1, OC_2, OC_3 are in GP.

- If PT_1, PT_2, PT_3 be the lengths of the tangents drawn from $P(x_1, y_1)$ to the three circles $x^2 + y^2 + 2h_i x - c^2 = 0 (i = 1, 2, 3)$, then PT_1, PT_2, PT_3 are in GP.
- Pole of $lx + my + n = 0$ w.r.t. $x^2 + y^2 = r^2$ is $\left(-\frac{r^2 l}{n}, -\frac{r^2 m}{n}\right)$
- The diameter corresponding to a system of parallel chords of a circle always passes through the centre of circle and is perpendicular to parallel chords.
- Let $S_1 = 0$, $S_2 = 0$ be the two circles with radii r_1, r_2 and $\frac{S_1}{r_1} \pm \frac{S_2}{r_2} = 0$ will meet at right angles.
- The angle between the two tangents from (α, β) to the circle $x^2 + y^2 = r^2$ is $2\tan^{-1}\frac{r}{\sqrt{S_1}}$, where $S_1 = x_1^2 + y_1^2 - r^2$.
- If two given circles intersect each other, then the radical axis is the same as the common tangent.
- The pair of tangents from (0, 0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at right angles, if $g^2 + f^2 = 2c$
- If circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the x-axis, then $g^2 = c$.
- If circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the y-axis, then $f^2 = c$.
- If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches both the axes, then $g^2 = f^2$.
- A line intersects a given circle at two distinct points, if the length of the perpendicular from the centre is less than the radius of the circle.
- A line touches a circle, if the length of the perpendicular from the centre is equal to the radius of the circle.
- The diameter corresponding to a system of parallel chords of a circle always passes through the centre of the circle and is perpendicular to the parallel chords.
- The circle $x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$ bisects the circumference of the second circle
$$x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0, \text{ if}$$
$$2g_2(g_1 - g_2) + 2f_2(f_1 - f_2) = c_1 - c_2.$$

Parabola

- Equation of the chord joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is $(t_1 + t_2)y = 2x + 2at_1t_2$.
- For PQ to be focal chord $t_1t_2 = -1$.
- Length of focal chord having 't_1' and 't_2' as end points is $a(t_2 - t_1)^2$.
- Angle between tangents at two points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$
- If the normals at 't_1' and 't_2' meet the parabola, then $t_1t_2 = 2$.
- The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.
- If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$.
- If the normal at the point 't_1' meets the parabola $y^2 = 4ax$ again at the points 't_2', then $t_2 = -t_1 - \frac{2}{t_1}$.
- Tangents drawn at the ends of any focal chord intersect at directrix of the parabola.
- Circle drawn at the ends of any focal chord touches directrix of the parabola.
- Normals drawn from a point $(h, 0)$ on the axis of the parabola $y^2 = 4ax$ are real (all three), if $h \ge 2a$
- $y = ax^2 + bx + c$ is a parabola with it's axis parallel to y-axis and it's vertex is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, where $D = b^2 - 4ac$.
- $x = ly^2 + my + n$ is a parabola with it's axis parallel to x-axis and it's vertex is $\left(\frac{4nl - m^2}{4l}, -\frac{m}{2l}\right)$.
- If a ray parallel to axis of parabola gets reflected on parabolic mirror, then reflected ray passes through it's focus.

- If a chord of the parabola $y^2 = 4ax$ subtends 90° angle at the vertex of the parabola, then $t_1t_2 = -4$ (where t_1 and t_2 are ends of the chord)
- Foot of the normals drawn from a point are called conormal points.

- Chords parallel to latus rectum of the parabola $y^2 = 4ax$ are called double ordinates.
- If the distance between directrix and latus rectum is $2a$, then length of latus rectum is $4a$.
- Equation of tangent to $(y-k)^2 = 4a(x-h)$ in slope form is $(y-k) = m(x-h) + \frac{a}{m}$.
- The locus of the points of intersection of perpendicular tangents to the parabola $y^2 = 4ax$ is its directrix.
- An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose vertices are at the parabola, then the length of its side is equal to $8a\sqrt{3}$.
- The angle of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is given by
$$\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}.$$
- Equation of common tangent to the parabolas $x^2 = 4by$ and $y^2 = 4ax$ is $yb^{1/3} + xa^{1/3} + (ab)^{2/3} = 0$.
- The point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $(0, 0)$ and $(4a, 4a)$.
- The length of the intercept made by the line $y = mx + c$ on the parabola $y^2 = 4ax$ is $\frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$.
- If the chord joining the points having parameters t_1 and t_2 passes through the focus, then $t_1t_2 = -1$.
- For the ends of latus rectum of the parabola $y^2 = 4ax$, the values of the parameter are ± 1.
- The area of the triangle formed by the three points on a parabola is twice the area of the triangle formed by the tangents at these points.
- The circle described on any focal chord of a parabola as diameter touches the directrix.
- The chord joining the points of contact of a pair of perpendicular tangents passes through [illegible] us of the parabola.
- Tangents drawn from any point on the directrix are at right angle.
- The pole of a focal chord of a parabola lies on its directrix.
- Every diameter of a parabola is parallel to its axis.
- The tangent at the point where a diameter meets the parabola is parallel to the system of chords bisected by the diameter.
- The tangents at the ends of any chord of parabola meet on the diameter which bisects the chord.
- The tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.
- The semi-latus rectum of a parabola is the harmonic mean between the segments of any focal chord of a parabola.
- The normals at the extremities of the latus rectum of the parabola intersect at right angles on the axis.
- If the tangent and normal at any point P of a parabola meet the axis in T and G respectively, then
- (i) $ST = SG = SP$, S being focus.
  (ii) The tangent at P is equally inclined to the axis and the focal distance.
- The area of the triangle formed by the tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ and their chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$
- The tangents at (x_1, y_1) and (x_2, y_2) to the parabola $y^2 = 4ax$ intersect at $\left(\frac{y_1y_2}{4a}, \frac{y_1+y_2}{2}\right)$.
- Semi-latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean of segments of any focal chord.
- The orthocentre of a triangle formed by any three tangents to a parabola lies on the directrix.

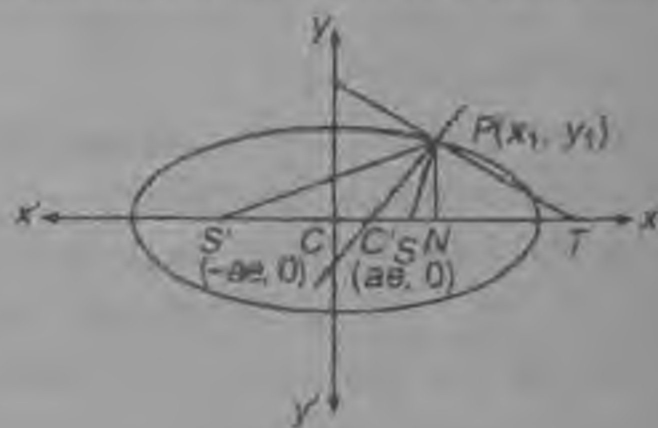
Ellipse

- If S be the focus and G be the point where the normal at P meets the axis of an ellipse, then $SG = eSP$ and the tangent and normal at P bisect the external and internal angles between the focal distance of P.
- Any tangent is the polar of its point of contact.
- The sum of focal distances of any points on the ellipse is equal to the major axis.
- Two tangents can be drawn from a point to an ellipse. The two tangent are real and distinct or coincidental or imaginary according as the given points lies outside or inside the ellipse.
- The product of perpendiculars from the foci on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to b^2.

- Locus of mid points of focal chord of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a^2}$.
- Locus of mid points of normal chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2.$$

- The area of parallelogram formed by tangents at the ends of conjugate diameters of an ellipse is constant and is equal to $4ab$.
- The normal at P on the ellipse with foci S and S' is internal bisector of $\Delta SPS'$.
- The centre of conic represented by $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be found by solving the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ and is equal to $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$.
- The locus of the point of intersection of mutually perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a circle.
- If any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts off intercepts of length h and k on the axes, then $\frac{a^2}{h^2}+\frac{b^2}{k^2}=1$.
- The line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.
- If the line $lx + my + n = 0$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $n^2=b^2m^2+a^2l^2$.
- From any point two tangents can be drawn to an ellipse which are real, coincident or imaginary according as the point is outside, on or inside the ellipse.
- The normal and tangent at point P of an ellipse bisect the internal and external angles between the focal distances of the point (see figure 22.16).



Hyperbola

- Two tangents can be drawn from a point to a hyperbola.
- The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$.
- Four normals can be drawn from a point to a hyperbola.
- Difference of the focal distances = $2a$.
- The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
- The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to its asymptotes is equal to $\frac{a^2b^2}{a^2+b^2}$.
- Locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a circle,

$$x^2+y^2=a^2-b^2.$$

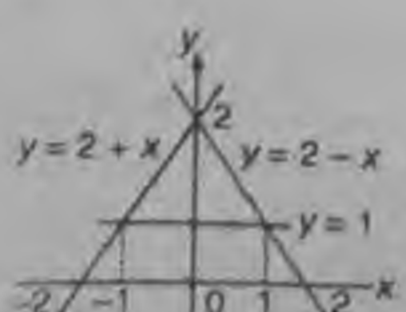
- Asymptotes always passes through the centre of the hyperbola.
- Centre of the hyperbola is the mid point of line joining two foci.
- The locus of the point of intersection of mutually perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a circle.
- The locus of the point of intersection of the lines $ax\sec\theta + by\tan\theta = a$ and $ax\tan\theta + by\sec\theta = b$, where θ is the parameter, is a hyperbola.
- The equation of a common tangent to the conics $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is $x+y=\sqrt{a^2-b^2}$.

- If the straight line $lx + my + n = 0$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.
- If the line $lx + my + n = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a^2l^2 - b^2m^2 = n^2$.
- The product of lengths of perpendiculars drawn from foci on any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is b^2.

Functions

- If a polynomial satisfies the property $f(x) + f\left(\frac{1}{x}\right) = f(x)\,f\left(\frac{1}{x}\right)$, then $f(x) = 1 \pm x^n$.
- Domain of polynomial functions is $(-\infty, \infty)$.
- Range of odd degree polynomial functions is $(-\infty, \infty)$.
- Even functions are many-one functions.
- Any function can be written as sum of an even function and an odd function.
  i.e., $f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$
- Every constant function is an even function and $f(x) = 0$ is an even function as well as an odd function.
- Function $f(x) = x$ is an identity function. ($D_f \in R$ and $R_f \in R$).
- Constant functions are periodic functions with no fundamental period.
- If $f(x)$ is inverse function of $g(x)$, then $f[g(x)] = x$ or $g[f(x)] = x$, i.e., same as $f^{-1}[f(x)] = x$ or $f[f^{-1}(x)] = x$.
- $a^{\log_a f(x)} = f(x)$
- $\log x^{2n} = 2n \log|x|$, where $n \in N$.
- $x + \frac{1}{x} \geq 2$ (If $x \in R^+$) and $x + \frac{1}{x} \leq -2$ (if $x \in R^-$)
- $f(x) = \min\{2 - x, 1, 2 + x\}$ means

$$f(x) = \begin{cases} 2 + x & x \leq -1 \\ 1 & -1 < x < 1 \\ 2 - x & x \geq 1 \end{cases} \qquad \text{(Clear from graph)}$$



- Two functions $y = f(x)$ and $y = g(x)$ can be combined (added, subtracted, multiplied or divided) only in common domains.
- Range of $f(x) = a \sin x + b \cos x + c$ is $[c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}]$
- Range of $f(x) = ax^2 + bx + c$ is $\left[-\frac{D}{4a}, \infty\right)$, if $a > 0$ and $\left(-\infty, -\frac{D}{4a}\right]$, if $a < 0$. (where $D = b^2 - 4ac$)
- LCM of $\frac{a}{b}$ and $\frac{c}{d} = \frac{\text{LCM of } a \text{ and } c}{\text{HCF of } b \text{ and } d}$.
- LCM of a rational number and an irrational number does not exist.
- If A and B are two distinct sets which have m and n elements respectively, then the number of mapping from A to $B = m^n$.
- If $A = B$, then number of mappings $= n^n$.
- If A and B are two distinct sets each of them have n elements, then number of bijective function $= n!$
- (a) If gof is one-one, then f is one-one.
  (b) If gof is onto, then f is onto.
- If $f(x)$ is a function, then
  (a) $f(x) + f(-x)$ is an even function.
  (b) $f(x) - f(-x)$ is an odd function.
- If $y = f(x) = \frac{ax + b}{x - a}$, then $(fof)(x) = x$
- If $f(x) + f\left(\frac{1}{x}\right) = f(x)\,f\left(\frac{1}{y}\right)$, then $f(x) = 1 \pm x^n$.
- $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π.
- $\tan x$, $\cot x$ are periodic functions with period π.
- $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\sec x|, |\operatorname{cosec} x|$ are periodic functions with period π.
- $\sin^n x$, $\cos^n x$, $\sec^n x$, $\operatorname{cosec}^n x$ are periodic functions with period 2π or π according n is odd or even.
- $\tan^n x$ and $\cot^n x$ are periodic functions with period π for all $n \in N$.
- $\sin^{-1} \sin x$, $\cos^{-1} \cos x$ are periodic functions with period 2π.
- $\tan^{-1} \tan x$ is periodic with period π.
- $\sin \sin^{-1} x$, $\cos \cos^{-1} x$, $\tan \tan^{-1} x$ are not periodic functions.

Limits, Continuity and Differentiability

- $\lim_{x \to 1} \cos^{-1} x$, $\lim_{x \to 0} \sqrt{x}$ does not exist because one side of limit can not be evaluated as $\cos^{-1}(1^+)$ is not defined and $\sqrt{0^-}$ is also not defined.
- $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist because we are not sure what value it takes in $[-1, 1]$, through it's a finite value.
- $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$, does not exist because it is not known what is the number just adjacent to any number, so we do not know just adjacent to a rational number or irrational number, number is rational or irrational.
- $\lim_{x \to 0^+} \frac{[x]}{x}$ is 0, here this is not $\frac{0}{0}$ form as Nr. is 0 and Dr. approaches 0. Similarly, $\lim_{x \to 0^+} ([x+1])^{1/x}$ is 1, this is not 1^∞ form as $\lim_{x \to 0^+} [x+1] = 1$ (exactly).
- $\lim_{x \to 0} e^{1/x}$ does not exist, as we have to find LHL and RHL separately as $1/x$ changes it's characteristics at 0 (discontinuous graph, rectangular hyperbola).

  Here LHL $= 0$ and RHL $= \infty$.
- If a function is continuous at $x = a$, then $\lim_{x \to a} f(x)$ exist, but converse may not be true.
- All polynomials, logarithmic functions, exponential functions, trigonometric functions, modulus function are continuous in their domains. Greatest Integer function is discontinuous at integers.
- If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then $\lim_{x \to a} [f(x)]^{g(x)} = l^m$
- $\lim_{x \to a} \log f(x) = \log\left(\lim_{x \to a} f(x)\right)$
- $\lim_{x \to a} fog(x) = f\left(\lim_{x \to a} g(x)\right)$
- If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$$

- $\lim_{x \to a} [1 + f(x)]^{1/g(x)} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$
- If $0 < x < y < z$, then
  $\lim_{n \to \infty} (x^n + y^n + z^n)^{1/n} = z$

Differentiation

- $\frac{d}{dx}[f(x)]^{f(x)} = [f(x)]^{f(x)}[1 + \log f(x)]\, f'(x)$
- If $y = [f(x)]^y$, then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)(1 - \log y)}$
- If $y = \sqrt{f(x) + y}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y - 1}$
- If $x^m y^n = (x+y)^{m+n}$, then $\frac{dy}{dx} = \frac{y}{x}$
- If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then

$$\frac{dy}{dx} = \frac{f'(x)}{g'(y)} \cdot \frac{1}{\sqrt{1 + f(x)^2}}$$

- If $e^{2g(y)} = \frac{1 + g(x)}{1 - g(x)}$, then

$$\frac{dy}{dx} = \frac{g'(x)}{g'(y)[1 - (g(x)^2)]}$$

- If $y = \sqrt{\frac{1 + g(x)}{1 - g(x)}}$, then

$$\frac{dy}{dx} = \frac{g'(x)}{(1 - g(x))^2}\sqrt{\frac{1 - g(x)}{1 + g(x)}}$$

- If $y = \cfrac{f(x)}{1 + \cfrac{g(x)}{1 + \cfrac{f(x)}{1 + \dots \infty}}}$, then

$$\frac{dy}{dx} = \frac{(1+y)f'(x) - yg'(x)}{1 + 2y + g(x) - f(x)}$$

- If $y = [f(x)]^{1/f(x)}$, then

$$\frac{dy}{dx} = \frac{y[1 - \log f(x)]\, f'(x)}{[f(x)]^2}$$

- If $y = \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y - 1}$
- If $[f(x)]^{g(y)} = e^{f(x) - g(y)}$, then

$$\frac{dy}{dx} = \frac{f'(x) \log f(x)}{g'(y)[1 + \log f(x)]^2}$$

- If $y = [f(x)]^{g(x)}$, then

$$\frac{dy}{dx} = y\left[\frac{g(x)\, f'(x)}{f(x)} + g'(x) \log f(x)\right]$$

- If $[f(x)]^{g(y)} = [g(y)]^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)}\left[\frac{f(x) \log g(y) - g(y)}{g(y) \log f(x) - f(x)}\right]$$

Applications of Derivatives

- If $f(x)$ in increasing, then $f^{-1}(x)$ is also increasing.
- If $f(x)$ is decreasing, then $f^{-1}(x)$ is also decreasing.
- If $f(x)$ and $g(x)$ are monotonic on $[a, b]$, then $g[f(x)]$ is also monotonic of same nature.
- If $y = f(x)$ is continuous in $[a, b]$ and $f(a)\, f(b) < 0$, then $y = f(x)$ intersect x-axis at least once.
- If $y = f(x)$ is continuous function and it's least value is m and greatest value is M, then $m \le f(x) \le M$ [Range of $y = f(x)$].
- If $x + y = c\ (x > 0, y > 0)$, then $xy \le \frac{c^2}{4}$, equality holds when $x = y \Rightarrow$ of in rectangles of a given perimeter, square has the largest area.
- If $xy = c^2\ (x > 0, y > 0)$, then $x + y \ge 2c$, equality holds when $x = y \Rightarrow$ in all the rectangles of a given area, square has the least perimeter.
- Rectangle of largest area inscribed in a given circle is a square whose length of diagonal is diameter of circle.
- If $f(x) = \frac{c}{g(x)}$ (where c is positive and independent of x) now, to find points of extreme for $f(x)$ first find points of extreme for $g(x)$ and points of maxima for $g(x)$ are points of minima for $f(x)$ and similarly points of minima for $g(x)$ are points of maxima for $f(x)$. But $g(a)$ should not be zero such point is not in the domain of $f(x)$.
- If $f(x) = \sqrt{g(x)}\ [g(x) \ge 0]$ here the points where $g(x)$ is maximum $f(x)$ is also maximum and where $g(x)$ is minimum, $f(x)$ is also minimum.
- If $y = f(x)$ is an increasing and continuous function in $[a, b]$ (Domain $\in [a, b]$), then Range $\in [f(a), f(b)]$.
  Similarly, if $y = f(x)$ is a decreasing and continuous function in $[a, b]$, then Range $\in [f(b), f(a)]$.
- If $y = f(x)$ is an increasing and continuous function in (a, b) [Domain $\in (a, b)$], then Range $\in \left(\lim_{x \to a^+} f(x), \lim_{x \to b^-} f(x) \right)$.
  Similarly, if $y = f(x)$ is a decreasing and continuous function in (a, b), then Range $\in \left(\lim_{x \to a^-} f(x), \lim_{x \to b^+} f(x) \right)$
- Shortest distance between two curves is always along common normal.
- If $x = a$ is point of inflexion, then necessarily $f'(a) = 0$ and $f''(a) = 0$. But $f'(a) = 0$ and $f''(a) = 0$ not necessarily implies $x = a$ is point of inflexion.
- Geometrically, the tangent to the curve $y = f(x)$ at a point where the ordinate is maximum or minimum is parallel to the x-axis.
- Maxima and minima occur alternately.
- If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x between a and b, then $f(x)$ is necessarily the minimum and the least value.
- If y is maximum or minimum, then $\log y = z$ is also maxima or minimum provided $y > 0$.
- $y = f(x)$ is maximum or minimum according as $z = \frac{1}{f(x)}$ is minimum or maximum.
- If the sum of two positive numbers is constant, then their product is greatest when the numbers are equal.
- If product of two positive number is a constant, then their sum is least when the numbers are equal.
- The shortest distance between two curves lie along the common normal.
- The curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ cut orthogonally, if $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.
- $f(x) = \tan x - x,\ \forall\, x \in R$ is always increasing function.
- If at any point on the curve, the subtangent and subnormal are equal, then length of tangent is $\sqrt{2}$ times the ordinate.
- The function $f(\theta) = \sin^m \theta \cos^n \theta$ attains maximum value at $\theta = \tan^{-1} \sqrt{\frac{m}{n}}$.
- If AB is a diameter of a circle and C be any point on the circumference, then the area of $\Delta\, ABC$ will be maximum, if triangle is isosceles.

Indefinite Integration

- $\int [xf'(x) + f(x)]\, dx = x\, f(x) + c$
- $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$
- $\int (xe^x + e^x)\, dx = xe^x + c$
- $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

- When $a > b$,

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\right) + c$$

- When $a < b$, $\int \frac{dx}{a + b\cos x}$

$$= -\frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{a+b}}{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{a+b}}\right| + c$$

- When $a = b$, $\int \frac{dx}{a + b\cos x} = \frac{1}{2} a \tan\frac{x}{2} + c$
- When $a > b$,

$$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left\{\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}\right\} + c$$

- When $a < b$,

$$\int \frac{dx}{a + b\sin x}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{a\tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\frac{x}{2} + b + \sqrt{b^2 - a^2}}\right| + c$$

- When $a = b$,

$$\int \frac{dx}{a + b\sin x} = \frac{1}{a}[\tan x - \sec x] + c$$

- $\int \sin(\log x)\, dx = \frac{x}{2}[\sin(\log x) - \cos(\log x)] + c$
- $\int \cos(\log x)\, dx = \frac{x}{2}[\sin(\log x) + \cos(\log x)] + c$
- $\int \frac{1}{x^2(x-1)}\, dx = \int\left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\right) dx$
- $\int \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{a^2 - b^2}$

$$\left[\int \frac{1}{x^2 + b^2}\, dx - \int \frac{1}{x^2 + a^2}\, dx\right]$$

- $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \infty = \frac{\pi^2}{6}$
- $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty = \frac{\pi^2}{12}$
- $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \infty = \frac{\pi^2}{8}$
- $\frac{1}{2^2} + \frac{1}{4^2} \frac{1}{6^2} + \ldots \infty = \frac{\pi^2}{24}$
- $\sin^{-1}(\log x) + \cos^{-1}(\log x) \neq \frac{\pi}{2}$
- Since, $\sin^{-1} x$ and $\cos^{-1} x$ are defined for $[-1, 1]$ for

$$x > e \Rightarrow \log x > 1$$

Definite Integration

- If $f(x) \geq 0$ and $a < b$, then $\int_a^b f(x)\, dx \geq 0$,

e.g., $\int_0^{\pi/2} \sin x\, dx = 1$.

- If $f(x) \geq 0$ and $a < b$, then $\int_b^a f(x) dx \leq 0$,

e.g., $\int_{\pi/2}^0 \cos x\, dx = -1$.

- If $f(x) \leq 0$ and $a < b$, then $\int_a^b f(x) dx \leq 0$,

e.g., $\int_{\pi/2}^0 \cos x\, dx = -1$.

- $\int_0^x [x]\, dx = \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \ldots + \int_{[x]}^x [x]\, dx$,

where $[\cdot]$ denotes greatest integer of x.

- $\int_0^{\pi/2} \log(\sin x)\, dx = \int_0^{\pi/2} \log(\cos x) dx = -\frac{\pi}{2}\log 2$
- $\int_0^{\pi/2} \log(\tan x)\, dx = \int_0^{\pi/2} \log(\cot x)\, dx = 0$
- $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$
- $\int_0^a \frac{dx}{1 + e^{\sin x}} = \frac{\pi}{2}$
- $\int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx = \frac{a}{2}$
- $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}$
- $\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$
- $\int_0^{\pi/4} \tan^n x\, dx + \int_0^{\pi/4} \tan^{n-2} dx = \frac{1}{n-1}$
- If $I_n = \int_0^{\pi/2} \sin^n x\, dx$, then $I_n = \frac{n-1}{n} I_{n-2}$
- If $I_n = \int_0^{\pi/2} \cos^n x\, dx$, then $I_n = \frac{n-1}{n} I_{n-2}$
- $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$

$$= \int_0^a f(x)\, dx + \int_0^a f(a + x)\, dx$$

- $\int_a^b (x)\, dx = (b - a)\int_0^1 x\, dx$, where $(\cdot)$ denotes fractional part of x.

e.g., $\int_0^5 \{x\}\,dx = 5\int_0^1 x\,dx = \frac{5}{2}$

- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
- $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$
- $\int_0^{\pi/2} \cos^{n-2} \sin nx\,dx = \frac{1}{n-1}$, if $n > 1$ and $n \in I$
- If m, n are positive integers, then

  (a) $\int_0^{\pi/2} \sin nx \cos mx\,dx$

  $$= \begin{cases} \frac{2m}{m^2-n^2}, & \text{if } m-n \text{ is odd} \\ 0, & \text{if } m-n \text{ is even} \end{cases}$$

  (b) $\int_0^{\pi} \cos mx \sin nx\,dx$

  $$= \begin{cases} \frac{2m}{m^2-n^2}, & \text{if } n-m \text{ is odd} \\ 0, & \text{if } n-m \text{ is even} \end{cases}$$

- $\int_0^{\pi} \sin mx \sin nx\,dx = \begin{cases} \frac{\pi}{2}, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$
- $\int_0^{\pi} \cos mx (\cos x)^n\,dx = 0$, if $m > n$
- $\int_0^{\infty} e^{-ax} \cos bx\,dx = \frac{a}{a^2+b^2}$

  and $\int_0^{\infty} e^{-ax} \sin bx\,dx = \frac{b}{a^2+b^2}$
- $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \frac{\pi}{2}$
- $\int_{-\infty}^{\infty} \frac{x^c}{c^x}\,dx = \frac{\sqrt{c+1}}{(\log c)^{c+1}}$
- $\sum_{r=0}^{n-1} \sin(\alpha + r\beta) = \sin\left(\alpha + \frac{1}{2}(n-1)\beta\right) \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$
- $\sum_{r=0}^{n-1} \cos(\alpha + r\beta) = \cos\left(\alpha + \frac{1}{2}(n-1)\beta\right) \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$
- $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$
- $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

Area Bounded by the Curves

- The area of the region bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16\,ab}{3}$ sq unit.
- Area of the region bounded by $y^2 = 4ax$ and $y = mx$ is $\frac{8a^2}{3m^3}$ sq unit.
- Area of the region bounded by $y^2 = 4ax$ and its latus rectum $= \frac{8a^2}{3}$ sq unit.
- Area of the region bounded by one arch of $\sin ax$ or $\cos ax$ and x-axis is $\frac{2}{a}$ sq unit.
- Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq unit.
- Area bounded by $y = ax^2 + bx + c$ and x-axis is $\frac{(b^2-4ac)^{3/2}}{6a^2}$.
- $\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \cos x\,dx = 1$
- $\int_0^1 \{x\}\,dx = \frac{1}{2}$
- $\int_0^1 \log x\,dx = -1$

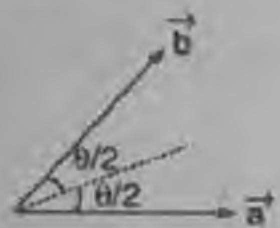
Differential Equation

- $d(x+y) = dx + dy$
- $d(xy) = y\,dx + x\,dy$
- $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$
- $d[\log(xy)] = \frac{y\,dx + x\,dy}{xy}$
- $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{xy} = \frac{dy}{y} - \frac{dx}{x}$
- $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{x\,dy - y\,dx}{x^2-y^2}$

- $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=\frac{x\,dy-y\,dx}{x^2+y^2}$
- $d(\sqrt{x^2+y^2})=\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}$
- $d(x^2+y^2)=dx^2+dy^2=2x\,dx+2y\,dy$
- $d\left(\frac{1}{y}-\frac{1}{x}\right)=d\left(\frac{1}{y}\right)-d\left(\frac{1}{x}\right)=\frac{dx}{x^2}-\frac{dy}{y^2}$

Vector Algebra

- The vector along the bisector of angle between vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where λ is a parameter.
- The unit vector along the bisector of angle between $\vec{a}$ and $\vec{b}$ is $\frac{\hat{a}+\hat{b}}{2\cos\frac{\theta}{2}}$, where θ is angle between $\vec{a}$ and $\vec{b}$.



- For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$$[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$$

- If diagonals of a parallelogram is along the vectors $\vec{a}$ and $\vec{b}$, then area of parallelogram $=\frac{1}{2}|\vec{a}\times\vec{b}|$.
- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are position vectors of A, B and C of ΔABC, then position vector of centroid G will be $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ and $\vec{GA}+\vec{GB}+\vec{GC}=0$ and position vector of incentre will be

$$\frac{|\vec{b}-\vec{c}|\,\vec{a}+|\vec{c}-\vec{a}|\,\vec{b}+|\vec{a}-\vec{b}|\,\vec{c}}{|\vec{a}-\vec{b}|+|\vec{b}-\vec{c}|+|\vec{c}-\vec{a}|}.$$

- If D, E and F are mid points of AB, BC and CA respectively of ΔABC, then $\vec{AD}+\vec{BE}+\vec{CF}=\vec{0}$.
- If circumcentre, centroid and orthocentre of triangle ABC are O, G and H, then

$$\vec{OA}+\vec{OB}+\vec{OC}=3\vec{OG}=\vec{OH}$$

$$\text{and }\ \vec{HA}+\vec{HB}+\vec{HC}=3\vec{HG}=3\vec{HO}$$

- $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$

$$=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}$$

- Area of quadrilateral $ABCD=\frac{1}{2}|\vec{AC}\times\vec{BD}|$, where $\vec{AC}$ and $\vec{BD}$ are diagonals of quadrilateral.
- Volume of tetrahedron : If A, B, C, D are vertices of tetrahedron $ABCD$, then its volume $=\frac{1}{6}[\vec{AB}\ \ \vec{AC}\ \ \vec{AD}]$.
- $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Leftrightarrow \vec{a}\perp\vec{b}$
- $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}| \Leftrightarrow \vec{a}\,||\,\vec{b}$
- $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2 \Leftrightarrow \vec{a}, \vec{b}$ are orthogonals.
- If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then
  1. $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$ is a zero vector.
  2. $\vec{a}\times(\vec{b}\times\vec{c}),\ \vec{b}\times(\vec{c}\times\vec{a}),\ \vec{c}\times(\vec{a}\times\vec{b})$ are coplanar vectors.
- If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then
  1. $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.
  2. $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are coplanar.
- $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$
- $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$

Three Dimensional Geometry

- Image (x, y, z) (or reflection) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$
- Foot (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$
- Two spheres touch each other, if distance between their centres C_1 and C_2 is $r_1 + r_2$. If $|C_1C_2| > r_1 + r_2$, then they do not touch and do not intersect. If $|C_1C_2| = |r_1 - r_2|$ they touch each other internally.
- If spheres S_1 and S_2 touch each other, then $S_1 - S_2 = 0$ is common tangent plane. If S_1 and S_2 intersect each other, then $S_1 - S_2 = 0$ is equation of plane of common circle.
- Any plane parallel to XY plane is $z =$ constant, similarly plane parallel to YZ plane is $x =$ constant and plane parallel to ZX plane is $y =$ constant. $x = 0$, $y = 0$ and $z = 0$ are respectively YZ, ZX and XY planes.
- Any plane parallel to X-axis is of the form $by + cz = d$.
- Intersection point of planes $x = a$, $y = b$ and $z = c$ is (a, b, c).
- Points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are on same side of plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same sign. If they are of opposite sign, then the points are on the opposite sides.
- Equation of plane parallel to planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ and equidistance from them is
$$ax + by + cz + \left(\frac{d_1 + d_2}{2}\right) = 0$$
- The equation of plane containing the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and parallel to the line $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$
- The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ lies in the plane $a_2x + b_2y + c_2z + d_2 = 0$, then
$$a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$$
and
$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
- If a plane makes the intercepts of length a, b, c on the axes OX, OY and OZ, then area of triangle ABC is $\frac{1}{2}\sqrt{(ab)^2 + (bc)^2 + (ca)^2}$ sq unit.
- If a straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets a plane $\vec{r} \cdot \vec{n} = 0$ in R, then the position vector of R is given by $\vec{a} - \left(\frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}\right)\vec{b}$.
- Equation of the plane which bisect the join of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ perpendicularly, is
$$x(x_1 - x_2) + y(y_1 - y_2) + z(z_1 - z_2) = \frac{1}{2}[(x_1^2 - x_2^2) + (y_1^2 - y_2^2) + (z_1^2 - z_2^2)]$$
- If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar, then
$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$$
- If $S_1 = 0$ and $S_2 = 0$ be two intersecting spheres, then equation of the plane of circle in which these two spheres intersect is given by $S_1 - S_2 = 0$ and equation of common tangent plane to the sphere $S_1 = 0$, $S_2 = 0$, if they touch each other is given by $S_1 - S_2 = 0$.
- Points A, B, C form a plane whose equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ which does not passes through origin, then distance from origin is $\frac{abc}{\sqrt{(ab)^2 + (bc)^2 + (ca)^2}}$.

Dynamics

- Rate of change of velocity with respect to t is known as acceleration. So, the graph of slope of the velocity is same as the graph of acceleration.
- Area under the velocity time graph during an interval of time is same as the displacement during that interval.
- Locus of vertex of trajectory is an ellipse whose equation is given by $x^2 + 4y^2 = \frac{2u^2 y}{g}$
- Locus of the focus of trajectory is a circle whose equation is given by $x^2 + y^2 = \frac{u^2}{4g^2}$
- Velocity at any point is equal to velocity to its fall from directrix to that point.

Statistics

- Algebraic sum of the deviations of a set of values from their arithmetic mean is zero.
- The sum of the squares of the deviations of a set of values is minimum when taken about mean.
- Mean deviation is least when taken from median.
- If each of n given observations is doubled, then mean is also doubled.
- Standard deviation of n natural numbers

$$\sigma = \left(\frac{1}{12}(n^2 - 1)\right)^{1/2}$$

- **Combined standard deviation** : Let A_1 and A_2 be two series having n_1 and n_2 observations respectively. Let their AM be $\bar{x}_1$ and $\bar{x}_2$, and standard deviations be σ_1 and σ_2. Then the combined standard deviation σ or σ_{12} of A_1 and A_2 is given by

$$\sigma_{12} \text{ or } \sigma = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$$

$$= \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

where $d_1 = (\bar{x}_1 - \bar{x}_{12})$, $d_2 = (\bar{x}_2 - \bar{x}_{12})$ and

$$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} \text{ is the combined mean.}$$

- Variance is independent of change of origin but not of scale.
- Coefficient of skewness $= \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$